E) BOUSSINESQ'S PROBLEM - the concentrated force acting on the elastic semi-space

E.1) The equations of BELTRAMI and MITCHELL.

It follows to obtain the partial derivative equations for the stress tensor σ in the particular case of an elastic, homogeneous, isotropic media. In the beginning, the next symmetric tensor is evaluated

$$S = -\left[\operatorname{grad}(\rho \vec{b}) + \operatorname{grad}^t(\rho \vec{b})\right] \quad , \tag{e1}$$

Using the equilibrium equation, it follows that

$$\operatorname{div}\sigma = -\rho \vec{b} \tag{e2}$$

Hence

$$S = \operatorname{grad}(\operatorname{div}\sigma) + \operatorname{grad}^t(\operatorname{div}\sigma) \quad . \tag{e3}$$

The HOOKE's reversed law gives

$$\varepsilon = \frac{1+\nu}{E}\sigma - \frac{\nu}{E}\Theta 1 \quad , \tag{e4}$$

so

$$\sigma = \frac{E}{1+\nu}\varepsilon + \frac{\nu}{1+\nu}\Theta 1 \quad , \tag{e5}$$

where

$$\Theta = \operatorname{tr}\sigma \ , \ \vartheta = \operatorname{tr}\varepsilon = \frac{1-2\nu}{E}\Theta \quad . \tag{e6}$$

But

$$\operatorname{div}(f 1) = \operatorname{grad} f \quad . \tag{e7}$$

Hence

$$\operatorname{div}\sigma = \frac{E}{1+\nu}\operatorname{div}\varepsilon + \frac{\nu}{1+\nu}\operatorname{grad}\Theta \tag{e8}$$

Expression (e1) becomes

$$S = \frac{E}{1+\nu}\left[\operatorname{grad}(\operatorname{div}\varepsilon) + \operatorname{grad}^t(\operatorname{div}\varepsilon)\right] + \frac{\nu}{1+\nu}\left[\operatorname{grad}(\operatorname{grad}\Theta) + \operatorname{grad}^t(\operatorname{grad}\Theta)\right], \tag{e9}$$

Because the tensor $\operatorname{grad}(\operatorname{grad}\Theta)$ is a symmetric one, it follows that

$$S = \frac{E}{1+\nu}\left[\operatorname{grad}(\operatorname{div}\varepsilon) + \operatorname{grad}^t(\operatorname{div}\varepsilon)\right] + \frac{2\nu}{1+\nu}\operatorname{grad}(\operatorname{grad}\Theta) \ . \tag{e10}$$

The tensor in the first parenthesis of (e10) has the ij - component equal to

$$\begin{aligned}
&\left[\operatorname{grad}(\operatorname{div}\varepsilon) + \operatorname{grad}^t(\operatorname{div}\varepsilon)\right]_{ij} = (\operatorname{div}\varepsilon)_{i,j} + (\operatorname{div}\varepsilon)_{j,i} = \varepsilon_{iq,qj} + \varepsilon_{jq,qi} = \\
&\frac{1}{2}\left(u_{i,qqj} + u_{q,iqj} + u_{j,qqi} + u_{q,jqi}\right) = \frac{1}{2}\left[\left(u_{i,j} + u_{j,i}\right)_{,qq} + 2\left(u_{q,q}\right)_{,ij}\right] = \\
&\left(\Delta\varepsilon\right)_{ij} + \left(\operatorname{tr}\varepsilon\right)_{,ij} = \left(\Delta\varepsilon\right)_{ij} + \frac{1-2\nu}{E}\Theta_{,ij} = \left(\Delta\varepsilon\right)_{ij} + \frac{1-2\nu}{E}\left[\operatorname{grad}(\operatorname{grad}\Theta)\right]_{ij}
\end{aligned} \tag{e11}$$

Hence

$$\operatorname{grad}(\operatorname{div}\varepsilon) + \operatorname{grad}^t(\operatorname{div}\varepsilon) = \Delta\varepsilon + \frac{1-2\nu}{E}\operatorname{grad}(\operatorname{grad}\Theta) \tag{e12}$$

Using (e12) and (e10), the expression (e1) becomes

$$\mathbf{S} = \frac{E}{1+\nu}\Delta\boldsymbol{\varepsilon} + \frac{1}{1+\nu}\,\text{grad grad}\Theta \tag{e13}$$

Applying the 3-dimensional LAPLACE operator Δ in (e4), it follows

$$\Delta\boldsymbol{\varepsilon} = \frac{1+\nu}{E}\Delta\boldsymbol{\sigma} - \frac{\nu}{E}\Delta\Theta\mathbf{1} \tag{e14}$$

Replacing (e14) into (e13) gives

$$\mathbf{S} = \Delta\boldsymbol{\sigma} - \frac{\nu}{1+\nu}\Delta\Theta\mathbf{1} + \frac{1}{1+\nu}\,\text{grad grad}\Theta\ , \tag{e15}$$

i.e.

$$\Delta\boldsymbol{\sigma} - \frac{\nu}{1+\nu}\Delta\Theta\mathbf{1} + \frac{1}{1+\nu}\,\text{grad grad}\Theta = -\left[\text{grad}(\rho\,\vec{b}) + \text{grad}^{t}(\rho\,\vec{b})\right] \tag{e16}$$

But

$$\begin{aligned}
&\text{tr}\,(\Delta\boldsymbol{\sigma}) = \Delta(\text{tr}\,\boldsymbol{\sigma}) = \Delta\,\Theta\,,\\
&\text{tr}\,(\text{grad grad}\,\Theta) = \Delta\,\Theta\,,\\
&\text{tr}\,(\Delta\Theta\mathbf{1}) = 3\Delta\,\Theta\,,\\
&\text{tr}\left[\text{grad}(\rho\,\vec{b}) + \text{grad}^{t}(\rho\,\vec{b})\right] = 2\text{tr}\left[\text{grad}\,(\rho\,\vec{b})\right] = 2\,\text{div}(\rho\,\vec{b})
\end{aligned} \tag{e17}$$

Applying the trace operator in (e16) and using eqs.(e17) gives

$$\Delta\Theta + \frac{1}{1+\nu}\Delta\Theta - \frac{3\nu}{1+\nu}\Delta\Theta = -\,2\,\text{div}(\rho\,\vec{b}) \tag{e18}$$

It follows the trace of the stress tensor verifies the relation

$$\Delta\Theta = -\frac{1+\nu}{1-\nu}\,\text{div}(\rho\,\vec{b}) \tag{e19}$$

Replacing (e19) into (e16) gives

$$\Delta\boldsymbol{\sigma} + \frac{\nu}{1-\nu}\,\text{div}(\rho\,\vec{b})\mathbf{1} + \frac{1}{1+\nu}\,\text{grad grad}\Theta = -\left[\text{grad}(\rho\,\vec{b}) + \text{grad}^{t}(\rho\,\vec{b})\right] \tag{e20}$$

Eqs.(e19)-(e20) represents the equations of BELTRAMI and MITCHELL, having as unknowns only the elements of the stress tensor . Together with appropriate conditions (in tensions) on the boundary of the elastic body, they allow one to solve the corresponding linear static problem.

Particular cases.

a) Suppose that

$$\text{div}(\rho\,\vec{b}) = 0\ , \tag{e21}$$

i.e. a vector potential $\vec{\psi}$ exists having the property that

$$\rho\,\vec{b} = \text{rot}\,\vec{\psi}\ , \tag{e22}$$

From (e19) it follows that the traces of both stress and strain tensors are harmonic functions

$$\Delta\Theta = \Delta\theta = 0\ , \tag{e23}$$

b) Suppose that

$$\rho\,\vec{b} = \text{grad}\,\varphi\ ,\ \text{where}\ \ \Delta\varphi = 0\ , \tag{e24}$$

it follows that

$$\text{div}(\rho\,\vec{b}) = \Delta\varphi = 0\ , \tag{e25}$$

i.e. eq. (e23) is verified and eq.(20) gives

$$\Delta\boldsymbol{\sigma} + \frac{1}{1+\nu}\,\text{grad grad}\Theta = -\,2\text{grad}(\text{grad}\varphi) \tag{e26}$$

Applying the LAPLACE operator Δ to eq.(e26), it follows that the stress tensor is the solution of the þi-harmonic equation

$$\Delta\Delta\sigma = 0 \quad . \qquad \text{(e27)}$$

In most real cases, the volume forces are neglected (or they are represented only by the weight of the body, satisfying eq.(24). It follows the elastic linear problem involve solving harmonic and bi-harmonic equations.

E.2) The model.

A co-ordinate system having the third vertical axis positive downward is used. The semi-space $x_3 \geq 0$ is represented by an elastic, homogeneous, isotropic medium having the elastic coefficients λ and μ (or E and ν respectively). In the origin of the co-ordinate system is acting a vertical force having the magnitude equal to P. It follows to find the stress and the displacements. Spherical co-ordinates will be used (Fig.E1):

$$x_1 = r \sin\vartheta \cos\lambda\,,\ \ x_2 = r\sin\vartheta\sin\lambda\,,\ \ x_3 = r\cos\vartheta \qquad \text{(e28)}$$

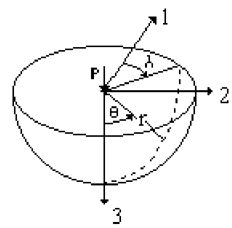

Fig.E1 The spherical co-ordinate system and a vertical force of magnitude equal to P acting at the origin.

Because symmetry, the displacement vector has the components like

$$u_r = u_r(r,\vartheta)\,,\ \ u_\lambda = 0\,,\ \ u_\vartheta = u_\vartheta(r,\vartheta) \qquad \text{(e29)}$$

It follows the components of both stress and strain tensors are functions of r and ϑ only.

E.3) The equations of equilibrium and strain tensor in spherical co-ordinates.

The LAMÉ differential parameters for spherical co-ordinates are

$$h^1 = 1\,,\ \ h^2 = r\,,\ \ h^3 = r\sin\vartheta \qquad \text{(e30)}$$

The generalised curvilinear co-ordinates are

$$c^1 = r\,,\ \ c^2 = \vartheta\,,\ \ c^3 = \lambda \qquad \text{(e31)}$$

The unit vectors of the axis are

$$\vec{n}^1 = \vec{e}_r\ ,\ \ \vec{n}^2 = \vec{e}_\vartheta\ ,\ \ \vec{n}^3 = \vec{e}_\lambda \qquad \text{(e32)}$$

By using, for example, (IVAN, 1996), the divergence of a symmetric tensor T in spherical co-ordinates is

$$\begin{aligned}
\operatorname{div} T = \\
&\left(\frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{r\vartheta}}{\partial\vartheta} + \frac{1}{r\sin\vartheta}\frac{\partial T_{r\lambda}}{\partial\lambda} + \frac{2T_{rr}}{r} + \frac{T_{r\vartheta}}{r\tan\vartheta} - \frac{T_{\vartheta\vartheta}+T_{\lambda\lambda}}{r}\right)\vec{e}_r + \\
&\left(\frac{\partial T_{r\vartheta}}{\partial r} + \frac{1}{r}\frac{\partial T_{\vartheta\vartheta}}{\partial\vartheta} + \frac{1}{r\sin\vartheta}\frac{\partial T_{\vartheta\lambda}}{\partial\lambda} + \frac{3T_{r\vartheta}}{r} + \frac{T_{r\vartheta}}{r\tan\vartheta} + \frac{T_{\vartheta\vartheta}-T_{\lambda\lambda}}{r\tan\vartheta}\right)\vec{e}_\vartheta + \qquad \text{(e33)} \\
&\left(\frac{\partial T_{r\lambda}}{\partial r} + \frac{1}{r}\frac{\partial T_{\vartheta\lambda}}{\partial\vartheta} + \frac{1}{r\sin\vartheta}\frac{\partial T_{\lambda\lambda}}{\partial\lambda} + \frac{3T_{r\lambda}}{r} + \frac{2T_{\vartheta\lambda}}{r\tan\vartheta}\right)\vec{e}_\lambda
\end{aligned}$$

In the same way, the gradient of a vector in spherical co-ordinates is

$$\operatorname{grad}\vec{v}=\frac{\partial v_r}{\partial r}\vec{e}_r\otimes\vec{e}_r+\left(\frac{1}{r}\frac{\partial v_r}{\partial\vartheta}-\frac{v_\vartheta}{r}\right)\vec{e}_r\otimes\vec{e}_\vartheta+\left(\frac{1}{r\sin\vartheta}\frac{\partial v_r}{\partial\lambda}-\frac{v_\lambda}{r}\right)\vec{e}_r\otimes\vec{e}_\lambda+$$
$$\frac{\partial v_\vartheta}{\partial r}\vec{e}_\vartheta\otimes\vec{e}_r+\left(\frac{1}{r}\frac{\partial v_\vartheta}{\partial\vartheta}+\frac{v_r}{r}\right)\vec{e}_\vartheta\otimes\vec{e}_\vartheta+\left(\frac{1}{r\sin\vartheta}\frac{\partial v_\vartheta}{\partial\lambda}-\frac{v_\lambda}{r\tan\vartheta}\right)\vec{e}_\vartheta\otimes\vec{e}_\lambda+\qquad\text{(e34)}$$
$$\frac{\partial v_\lambda}{\partial r}\vec{e}_\lambda\otimes\vec{e}_r+\frac{1}{r}\frac{\partial v_\lambda}{\partial\vartheta}\vec{e}_\lambda\otimes\vec{e}_\vartheta+\left(\frac{1}{r\sin\vartheta}\frac{\partial v_\lambda}{\partial\lambda}+\frac{v_r}{r}+\frac{v_\vartheta}{r\tan\vartheta}\right)\vec{e}_\lambda\otimes\vec{e}_\lambda$$

The components of the strain tensor are

$$\varepsilon_{rr}=\frac{\partial u_r}{\partial r},\ \varepsilon_{r\vartheta}=\frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial\vartheta}-\frac{u_\vartheta}{r}+\frac{\partial u_\vartheta}{\partial r}\right),$$
$$\varepsilon_{r\lambda}=\frac{1}{2}\left(\frac{1}{r\sin\vartheta}\frac{\partial u_r}{\partial\lambda}-\frac{u_\lambda}{r}+\frac{\partial u_\lambda}{\partial r}\right),\varepsilon_{\vartheta\vartheta}=\frac{1}{r}\frac{\partial u_\vartheta}{\partial\vartheta}+\frac{u_r}{r},\qquad\text{(e35)}$$
$$\varepsilon_{\vartheta\lambda}=\frac{1}{2}\left(\frac{1}{r\sin\vartheta}\frac{\partial u_\vartheta}{\partial\lambda}-\frac{u_\lambda}{r\tan\vartheta}+\frac{1}{r}\frac{\partial u_\lambda}{\partial\vartheta}\right),\varepsilon_{\lambda\lambda}=\frac{1}{r\sin\vartheta}\frac{\partial u_\lambda}{\partial\lambda}+\frac{u_r}{r}+\frac{u_\vartheta}{r\tan\vartheta}$$

In the particular case of eqs.(e29), it follows

$$\varepsilon_{rr}=\frac{\partial u_r}{\partial r},\quad \varepsilon_{r\vartheta}=\frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial\vartheta}-\frac{u_\vartheta}{r}+\frac{\partial u_\vartheta}{\partial r}\right),\quad \varepsilon_{r\lambda}=0$$
$$\varepsilon_{\vartheta\vartheta}=\frac{1}{r}\frac{\partial u_\vartheta}{\partial\vartheta}+\frac{u_r}{r},\quad \varepsilon_{\vartheta\lambda}=0,\quad \varepsilon_{\lambda\lambda}=\frac{u_r}{r}+\frac{u_\vartheta}{r\tan\vartheta}\qquad\text{(e36)}$$

Hence

$$\sigma_{r\lambda}=\sigma_{\vartheta\lambda}=0\ ,\qquad\text{(e37)}$$

By using (e33), the equilibrium equations in the absence of the volume forces are

$$\frac{\partial\sigma_{rr}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{r\vartheta}}{\partial\vartheta}+\frac{3\sigma_{rr}}{r}+\frac{\sigma_{r\vartheta}}{r\tan\vartheta}=\frac{\sigma_{rr}+\sigma_{\vartheta\vartheta}+\sigma_{\lambda\lambda}}{r}\ ,\qquad\text{(e38)}$$
$$\frac{\partial\sigma_{r\vartheta}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{\vartheta\vartheta}}{\partial\vartheta}+\frac{3\sigma_{r\vartheta}}{r}+\frac{\sigma_{r\vartheta}}{r\tan\vartheta}+\frac{\sigma_{\vartheta\vartheta}-\sigma_{\lambda\lambda}}{r\tan\vartheta}=0$$

i.e.

$$\frac{\partial}{\partial r}\left(r^3\sin\vartheta\,\sigma_{rr}\right)+\frac{\partial}{\partial\vartheta}\left(r^2\sin\vartheta\,\sigma_{r\vartheta}\right)=r^2\sin\vartheta\,\Theta,$$
$$\frac{\partial}{\partial r}\left(r^3\sin\vartheta\,\sigma_{r\vartheta}\right)+\frac{\partial}{\partial\vartheta}\left(r^2\sin\vartheta\,\sigma_{\vartheta\vartheta}\right)=r^2\cos\vartheta\,\sigma_{\lambda\lambda}\qquad\text{(e39)}$$

E.4) LAPLACE operator in spherical co-ordinates. LEGENDRE's polynomials.

Using, for example, (IVAN, 1996) the gradient of a scalar function in spherical co-ordinates is

$$\operatorname{grad} f=\frac{\partial f}{\partial r}\vec{e}_r+\frac{1}{r}\frac{\partial f}{\partial\vartheta}\vec{e}_\vartheta+\frac{1}{r\sin\Delta}\frac{\partial f}{\partial\lambda}\vec{e}_\lambda\ .\qquad\text{(e40)}$$

Also, the LAPLACE operator is

$$\Delta f=\operatorname{div}\operatorname{grad} f=\frac{1}{r^2\sin\vartheta}\left[\sin\vartheta\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)+\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial f}{\partial\vartheta}\right)+\frac{1}{\sin\vartheta}\frac{\partial^2 f}{\partial\lambda^2}\right]\qquad\text{(e41)}$$

Consider the LAPLACE equation

$$\Delta f = 0 \tag{e42}$$

for the particular case when the unknown function has the form $f = f(r, \vartheta)$. Equation (e42) becomes

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{\sin\vartheta}\frac{\partial}{\partial \vartheta}\left(\sin\vartheta \frac{\partial f}{\partial \vartheta}\right) = 0 . \tag{e43}$$

By using the method of separation of the variables, a solution for eq.(43) has the form

$$f(r, \vartheta) = R(r)\, Y(\vartheta) , \tag{e44}$$

where R and Y are two unknown functions. Eq.(e43) becomes

$$\frac{\dfrac{d}{dr}\left(r^2 \dfrac{dR}{dr}\right)}{R} = -\frac{\dfrac{d}{d\vartheta}\left(\sin\vartheta \dfrac{dY}{d\vartheta}\right)}{Y \sin\vartheta} = k , \tag{e45}$$

where k is a constant. From (e45) it follows that

$$r^2 \frac{d^2 R}{dr^2} + 2r \frac{dR}{dr} - kR = 0 \tag{e46}$$

In a general case, the function R can be developed in power series. Let's look for a particular solution having the form $R_n(r) = r^n$ where n is a natural number. It follows from (e46) that

$$k = n(n+1) \tag{e47}$$

Then the particular solution of eq. (e46) can be expressed with the aid of two arbitrary constants as

$$R_n(r) = A_n\, r^n + B_n \,/\, r^n \tag{e48}$$

Because f has to approach finite values for $r \to \infty$, it has to take $A_n = 0$.
The second relation (e45) gives

$$\sin\vartheta \frac{d^2 Y}{d\vartheta^2} + \cos\vartheta \frac{dY}{d\vartheta} + n(n+1) Y \sin\vartheta = 0 \tag{e49}$$

By performing the substitution $z = \cos\vartheta$ eq.(e49) becomes

$$\frac{d}{dz}\left[\left(1 - z^2\right)\frac{dY}{dz}\right] + n(n+1)Y = 0 , \tag{e50}$$

The solution of (e50) is represented by the LEGENDRE polynomials denoted by $P_n(z)$, $n = 0,1,2,\ldots$ So,

$$P_0(z) = 1 \quad , \quad P_1(z) = z \quad , \quad P_2(z) = (3z^2 - 1)/2 \tag{e51}$$

Because the trace Θ of the stress tensor is a solution of the harmonic equation (e23), it follows that the general solution for that trace in the case of the BOUSSINESQ problem is

$$\Theta = \sum_{n=0}^{\infty} \frac{a_n}{r^{n+1}} P_n(\cos\vartheta) . \tag{e52}$$

According to eq.(e6), a similar solution exists for the trace of strain tensor.

E.5) The displacement field.

We look for a displacement field having the form

$$u_r = \frac{1}{r}\varphi(\vartheta) \quad , \quad u_\vartheta = \frac{1}{r}\psi(\vartheta) \quad , \quad u_\lambda = 0 \quad , \tag{e53}$$

where φ and ψ are two unknown functions following to be obtained. Substituting (e53) into (e36) it follows that

$$\begin{gathered} \varepsilon_{rr} = -\varphi / r^2 \ , \ \varepsilon_{r\vartheta} = (d\varphi / d\vartheta - 2\psi) / \left(2r^2\right) \ , \ \varepsilon_{r\lambda} = 0 \\ \varepsilon_{\vartheta\vartheta} = (\varphi + d\psi / d\vartheta) / r^2 \ , \ \varepsilon_{\vartheta\lambda} = 0 \ , \ \varepsilon_{\lambda\lambda} = (\varphi + \psi / \mathrm{tg}\vartheta) / r^2 \end{gathered} \tag{e54}$$

It follows that the trace of the strain tensor is

$$\theta = \mathrm{tr}\,\varepsilon = \varepsilon_{rr} + \varepsilon_{\vartheta\vartheta} + \varepsilon_{\lambda\lambda} = (\varphi + d\psi / d\vartheta + \psi / \mathrm{tg}\vartheta) / r^2 \tag{e55}$$

A comparison of (e55) to (e52) shows that

$$\varphi + d\psi / d\vartheta + \psi / tg\vartheta = a\cos\vartheta \ , \tag{e56}$$

where a is a constant remaining to be obtained. Hence

$$\theta = a\cos\vartheta / r^2 \ , \ \Theta = (3\lambda + 2\mu)\theta = (3\lambda + 2\mu)\, a\cos\vartheta / r^2 \ ,$$
$$\varepsilon_{\lambda\lambda} = \theta - (\varepsilon_{rr} + \varepsilon_{\vartheta\vartheta}) = (a\cos\vartheta - d\psi / d\vartheta) / r^2 \tag{e57}$$

By using HOOKE's law, it follows that

$$\sigma_{rr} = (\lambda\, a\cos\vartheta - 2\mu\varphi) / r^2 \ , \ \sigma_{r\vartheta} = \mu(d\varphi / d\vartheta - 2\psi) / r^2 \ ,$$
$$\sigma_{\vartheta\vartheta} = [\lambda\, a\cos\vartheta + 2\mu(\varphi + d\psi / d\vartheta)] / r^2 \ , \tag{e58}$$
$$\sigma_{\lambda\lambda} = [(\lambda + 2\mu)\, a\cos\vartheta - 2\mu d\psi / d\vartheta] / r^2$$

Substituting (e58) into the first equilibrium equation (e39) and using (e56), it follows after elementary computations that

$$\frac{d}{d\vartheta}\left(\sin\vartheta \frac{d\varphi}{d\vartheta}\right) = a\,(2 + \lambda / \mu)\sin 2\vartheta \ . \tag{e59}$$

Hence

$$\sin\vartheta \frac{d\varphi}{d\vartheta} = -\frac{a}{2}(2 + \lambda / \mu)\cos 2\vartheta + b = b - \frac{a}{2}(2 + \lambda / \mu) + a\,(2 + \lambda / \mu)\sin^2\vartheta \ ,$$
$$\frac{d\varphi}{d\vartheta} = \frac{b - a\,(2 + \lambda / \mu) / 2}{\sin\vartheta} + a\,(2 + \lambda / \mu)\sin\vartheta \tag{e60}$$

where b is an integration constant. Because

$$\int \frac{dx}{\sin x} = \ln\left| \ tg\frac{x}{2} \right| + C \ , \tag{e61}$$

where C is a new integration constant, it follows that

$$\varphi = \left[b - \frac{a}{2}(2 + \lambda / \mu)\right]\ln\left| \ tg\frac{\vartheta}{2} \right| - a\,(2 + \lambda / \mu)\cos\vartheta + C \ . \tag{e62}$$

For $\vartheta \to \pi / 4$, the logarithmic term into (e62) leads to infinite radial displacements. That can be avoided by taking

$$b - \frac{a}{2}(2 + \lambda / \mu) = 0 \ . \tag{e63}$$

Hence

$$\varphi = -a\,(2 + \lambda / \mu)\cos\vartheta + C \quad , \quad \frac{d\varphi}{d\vartheta} = a\,(2 + \lambda / \mu)\sin\vartheta \ . \tag{e64}$$

and

$$\sigma_{rr} = [(3\lambda + 4\mu)a\cos\vartheta - 2\mu C] / r^2 \ . \tag{e65}$$

Substituting eq.(e64) into eq.(e56), it follows that

$$\frac{d\psi}{d\vartheta} + \frac{\psi}{tg\vartheta} = a\,(3 + \lambda / \mu)\cos\vartheta + C \quad ,$$
$$\frac{d}{d\vartheta}(\psi\sin\vartheta) = \frac{a}{2}(3 + \lambda / \mu)\sin 2\vartheta - C\sin\vartheta \quad ,$$
$$\psi\sin\vartheta = -\frac{a}{4}(3 + \lambda / \mu)\cos 2\vartheta + C\cos\vartheta + D$$
$$= D - \frac{a}{4}(3 + \lambda / \mu) + \frac{a}{2}(3 + \lambda / \mu)\sin^2\vartheta + C\cos\vartheta \tag{e66}$$

Hence

$$\psi = \frac{D - a\,(3 + \lambda / \mu) / 4}{\sin\vartheta} + \frac{a}{2}(3 + \lambda / \mu)\sin\vartheta + C\ ctg\vartheta \ . \tag{e67}$$

Because the tangential displacements has to be finite ones for $\Delta \to 0$, it follows that

$$\psi = C\frac{\cos\vartheta - 1}{\sin\vartheta} + \frac{a}{2}(3+\lambda/\mu)\sin\vartheta = -C\,\mathrm{tg}\frac{\vartheta}{2} + \frac{a}{2}(3+\lambda/\mu)\sin\vartheta\,,$$
$$\frac{d\psi}{d\vartheta} = -\frac{C}{2\cos^2(\vartheta/2)} + \frac{a}{2}(3+\lambda/\mu)\cos\vartheta \qquad \text{(e68)}$$

By substituting eqs.(e64) and (e68) into eq.(e58) it follows that

$$\sigma_{r\vartheta} = \mu\left(2C\,\mathrm{tg}\frac{\vartheta}{2} - a\sin\vartheta\right)/\,r^2 \quad , \quad \sigma_{\vartheta\vartheta} = \mu\left[C\left(1-\mathrm{tg}^2\frac{\vartheta}{2}\right) - a\cos\vartheta\right]/\,r^2,$$
$$\sigma_{\lambda\lambda} = \mu\left[C\left(1+\mathrm{tg}^2\frac{\vartheta}{2}\right) - a\cos\vartheta\right]/\,r^2 \qquad \text{(e69)}$$

Substituting eqs.(e69) into the second equilibrium equation (e39) it can be seen that the last one is identically verified.

E.6) Boundary conditions for stress elements. The final solutions.

Eqs.(e65) and (e69) contain the unknown coefficients a and C. These constants follow to be obtained taking into account that the force P concentrated in the origin of the co-ordinate system is acting on the elastic semi-space. It can be seen that the points of the horizontal plane $x_3 = 0$ have the co-latitude $\vartheta = \pi/2$.

The unit vectors of the spherical co-ordinate system are related to the same vectors of the rectangular co-ordinate system by

$$\begin{pmatrix}\vec{e}_r\\ \vec{e}_\vartheta\\ \vec{e}_\lambda\end{pmatrix} = Q\begin{pmatrix}\vec{e}_1\\ \vec{e}_2\\ \vec{e}_3\end{pmatrix} \quad , \quad \begin{pmatrix}\vec{e}_1\\ \vec{e}_2\\ \vec{e}_3\end{pmatrix} = Q^t\begin{pmatrix}\vec{e}_r\\ \vec{e}_\vartheta\\ \vec{e}_\lambda\end{pmatrix} \quad , \qquad \text{(e70)}$$

where the orthogonal matrix is

$$Q = \begin{pmatrix}\sin\vartheta\cos\lambda & \sin\vartheta\sin\lambda & \cos\vartheta\\ \cos\vartheta\cos\lambda & \cos\vartheta\sin\lambda & -\sin\vartheta\\ -\sin\lambda & \cos\lambda & 0\end{pmatrix} . \qquad \text{(e71)}$$

The outer pointing unit vector normal to the elastic semi-space is equal to

$$-\vec{e}_3 = -\cos\vartheta\,\vec{e}_r + \sin\vartheta\,\vec{e}_\vartheta\,. \qquad \text{(e72)}$$

The resulting exterior force acting on the elastic semi-space is vanishing for all the points of the horizontal plane $\vartheta = \pi/2$, excepting the origin, i.e.

$$\sigma\left(-\vec{e}_3\right) = \vec{0}\,. \qquad \text{(e73)}$$

Substituting (e72) into (e73) for $\vartheta = \pi/2$ gives

$$\sigma_{\vartheta\vartheta} = 0 \quad , \quad \sigma_{r\vartheta} = 0 \quad . \qquad \text{(e74)}$$

By using (e69), the first eq.(e74) becomes an identity and the second one leads to

$$a = 2C\,. \qquad \text{(e75)}$$

So, eqs.(e65) and (e69) give

$$\sigma_{rr} = 2C\left[(3\lambda+4\mu)\cos\vartheta - \mu\right]/\,r^2\,, \qquad \text{(e76)}$$

$$\sigma_{r\vartheta} = -2C\mu\,\mathrm{tg}\frac{\vartheta}{2}\cos\Delta/\,r^2 \quad , \quad \sigma_{\vartheta\vartheta} = C\mu\cos\vartheta\left(\mathrm{tg}^2\frac{\vartheta}{2} - 1\right)/\,r^2. \qquad \text{(e77)}$$

Consider an elastic hemisphere having the centre at the origin of the co-ordinate system. The curved surface S of the hemisphere has the outer pointing normal equal to $\vec{e}_r$. On that surface, the rest of the elastic body (i.e. the semi-space minus the hemisphere) is acting on the hemisphere with a total force equal to

$$\iint_S \sigma\ \vec{e}_r\, dA = \iint_S \left(\sigma_{rr} \vec{e}_r + \sigma_{r\vartheta} \vec{e}_\vartheta \right) dA \ , \qquad \text{(e78)}$$

where dA is the surface element and the unit vectors are obtained with (e70). It follows that

$$\begin{aligned} \iint_S \sigma\ \vec{e}_r\, dA &= \int_0^{2\pi} d\lambda \int_0^{\pi/2} \sigma\ \vec{e}_r\, r^2 \sin\vartheta d\vartheta \\ &= 4\pi C \int_0^{\pi/2} \left\{ \left[(3\lambda + 4\mu)\cos\vartheta - \mu \right] \cos\vartheta \sin\vartheta + \mu\, \mathrm{tg}\frac{\vartheta}{2} \cos\vartheta \sin^2\vartheta \right\} d\vartheta\, \vec{e}_3 = 4\pi C(\lambda + \mu)\vec{e}_3 \end{aligned} \qquad \text{(e79)}$$

Because the hemisphere is into an equilibrium state, it follows that

$$\iint_S \sigma\ \vec{e}_r\, dA + P\vec{e}_3 = \vec{0} \ , \qquad \text{(e80)}$$

Hence

$$C = -\frac{P}{4\pi(\lambda + \mu)} \ . \qquad \text{(e81)}$$

Finally, the non-zero components of the stress tensor are equal to

$$\sigma_{rr} = -\frac{P}{2\pi(\lambda + \mu)} \left[(3\lambda + 4\mu)\cos\vartheta - \mu \right] / r^2 \ , \qquad \text{(e82)}$$

$$\sigma_{r\Delta} = \frac{P}{2\pi(\lambda + \mu)} \mu\, \mathrm{tg}\frac{\vartheta}{2} \cos\vartheta / r^2 \ , \qquad \text{(e83)}$$

$$\sigma_{\vartheta\vartheta} = \frac{P}{4\pi(\lambda + \mu)} \mu \cos\vartheta \left(1 - \mathrm{tg}^2 \frac{\vartheta}{2} \right) / r^2 . \qquad \text{(e84)}$$

The non-zero components of the displacement vector are

$$\begin{aligned} u_r &= \frac{P}{4\pi(\lambda + \mu)} \left[2(2 + \lambda/\mu)\cos\vartheta - 1 \right] / r \ , \\ u_\vartheta &= \frac{P}{4\pi(\lambda + \mu)} \left[\mathrm{tg}\frac{\vartheta}{2} - (3 + \lambda/\mu)\sin\vartheta \right] / r \end{aligned} \qquad \text{(e85)}$$

Using eq.(e71), the components of the stress tensor into the Cartesian base can be obtained as

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ & \sigma_{22} & \sigma_{23} \\ & & \sigma_{33} \end{pmatrix} = Q^t \begin{pmatrix} \sigma_{rr} & \sigma_{r\vartheta} & 0 \\ \sigma_{r\vartheta} & \sigma_{\vartheta\vartheta} & 0 \\ 0 & 0 & \sigma_{\lambda\lambda} \end{pmatrix} Q \qquad \text{(e86)}$$

Also, the components of the displacement vector into the Cartesian base are

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = Q^t \begin{pmatrix} u_r \\ u_\vartheta \\ 0 \end{pmatrix} \ . \qquad \text{(e87)}$$

Of particular importance in real life are the components

$$\sigma_{33} = -\frac{3P}{2\pi} z^3 / r^5 \quad , \quad u_3 = \frac{1+\nu}{E} \frac{P}{2\pi} \left[2(1-\nu)\frac{1}{r} + \frac{z^2}{r^3} \right] \ . \qquad \text{(e88)}$$

The BOUSSINESQ problem has a great importance in Geomechanics, in relation to the computation of a building foundation. The above solution derived for a concentrated vertical force can be used in the case of arbitrary vertical forces (spread on a certain domain) by assuming the principle of the superposition.